Lecture: 3-2 The Product and Quotient Rules

Example 1: How do we find the derivative of a product? Is it true that $(fg)' = f'g'$? Why or why not?

NO: $\frac{d}{dx} x^2 = 2x$; you know this.

But if you do: $\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$, which is wrong, thus this is a NON-RULE

BAD

The Product Rule: If f and g are differentiable then,

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Proof of why this is true:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

This is a clever trick!

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \boxed{- f(x+h)g(x) + f(x+h)g(x)} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)(g(x+h)-g(x))}{h} + \frac{g(x)(f(x+h)-f(x))}{h}\right)$$

$$= \lim_{h\to 0} f(x+h)\left(\frac{g(x+h)-g(x)}{h}\right) + \lim_{h\to 0} g(x)\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Example 2: If $f(x) = xe^x$ find $f'(x)$. Then find the second and third derivatives to find a formula for the nth derivative $f^{(n)}(x)$.

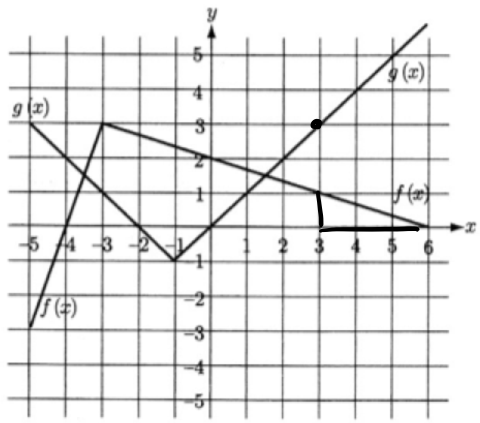
$$f'(x) = 1\cdot e^x + xe^x = \boxed{e^x + xe^x}$$

$$f''(x) = e^x + 1e^x + xe^x = \boxed{2e^x + xe^x}$$

$$f'''(x) = 2e^x + 1e^x + xe^x = \boxed{3e^x + xe^x}$$

$$\boxed{f^{(n)}(x) = ne^x + xe^x}$$

Example 3: If $h(x) = f(x)g(x)$ as shown below, find $h'(3)$.



$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(3) = f'(3)g(3) + f(3)g'(3)$

slope of f at x=3 function value of f at x=3

$= -\frac{1}{3}(3) + 1(1)$

$= -1 + 1$

$= \boxed{0}$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} = \frac{\text{lo de hi} - \text{hi de lo}}{\text{lo}^2}$$

Example 4: Find y' when $y = \dfrac{x^2 + x - 2}{x^3 + 6}$

simplify this

$$y' = \frac{(x^3+6)(2x+1) - (x^2+x-2)\cdot 3x^2}{(x^3+6)^2}$$

$$y' = \frac{2x^4 + x^3 + 12x + 6 - (3x^4 + 3x^3 - 6x^2)}{(x^3+6)^2}$$

$$\boxed{y' = \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}}$$

Example 5: Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$ and $g'(5) = 2$. Find the following values.

(a) $(f-g)'(5) = f'(5) - g'(5)$

$= 6 - 2$

$= \boxed{4}$

(b) $(fg)'(5)$ product rule!

$= f'(5)g(5) + f(5)g'(5)$

$= 6(-3) + 1(2)$

$= -18 + 2$

$= \boxed{-16}$

(c) $(g/f)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{(f(5))^2}$ quotient rule!

$= \dfrac{1(2) - (-3)(6)}{1^2}$

$= 2 + 18$

$= \boxed{20}$

Note: Don't use the product or quotient rule unless you have to. Here are a few examples where you can use the product or quotient rules, but it's easier not to!

Example 6: Find the derivative of the following functions.

(a) $f(t) = \sqrt{t}(2t+5)$ Distribute!

$$f(t) = 2t \cdot t^{1/2} + 5t^{1/2}$$
$$= 2t^{3/2} + 5t^{1/2}$$

$$f'(t) = 2 \cdot \frac{3}{2} t^{3/2-1} + 5 \cdot \frac{1}{2} t^{1/2-1}$$
$$= 3t^{1/2} + \frac{5}{2} t^{-1/2}$$
$$= 3\sqrt{t} \cdot \frac{2\sqrt{t}}{2\sqrt{t}} + \frac{5}{2\sqrt{t}}$$
$$= \boxed{\frac{6t+5}{2\sqrt{t}}}$$

(b) $y = \dfrac{t^3+t+5}{t^4} = (t^3+t+5)t^{-4}$ distribute and add exponentials

$$= t^{-1} + t^{-3} + 5t^{-4}$$
$$y' = -1t^{-2} - 3t^{-4} + 5(-4)t^{-5}$$
$$= -\frac{1}{t^2}\frac{t^3}{t^3} - \frac{3}{t^4}\frac{t}{t} - \frac{20}{t^5}$$
$$= \boxed{\frac{-t^3 - 3t - 20}{t^5}}$$

there's no way to simplify this

Example 7: Find the derivatives of the following functions

Distribute!

(a) $f(z) = (z^2 - \sqrt{z})(z^2 + \sqrt{z})$

$$= z^4 + z^2\sqrt{z} - z^2\sqrt{z} - z$$
$$= z^4 - z$$
$$\boxed{f'(z) = 4z^3 - 1}$$

this is WAY easier than using the product rule.

(b) $y = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

Don't do this: $\frac{\sqrt{x}}{\sqrt{x}} - \frac{1}{1} = 1 - 1 = 0$, because its wrong

$$y' = \frac{(\sqrt{x}+1)(\frac{1}{2}x^{-1/2}) - (\sqrt{x}-1)\cdot \frac{1}{2}x^{-1/2}}{(\sqrt{x}+1)^2}$$
$$= \frac{(\sqrt{x}+1)(\frac{1}{2\sqrt{x}}) - (\sqrt{x}-1)\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$
$$= \frac{\frac{1}{2\sqrt{x}}(\sqrt{x}+1-\sqrt{x}+1)}{(\sqrt{x}+1)^2}$$
$$= \frac{\frac{1}{2\sqrt{x}}(2)}{(\sqrt{x}+1)^2}$$
$$= \frac{\frac{1}{\sqrt{x}}\frac{\sqrt{x}}{1}}{(\sqrt{x}+1)^2\left(\frac{\sqrt{x}}{1}\right)} = \boxed{\frac{1}{\sqrt{x}(\sqrt{x}+1)^2}}$$

This one uses BOTH the product + quotient rules.

Example 8: Find the derivative of $f(x) = \dfrac{xe^x}{x + e^x}$.

$$f'(x) = \frac{(x+e^x)\frac{d}{dx}(xe^x) - xe^x \frac{d}{dx}(x+e^x)}{(x+e^x)^2}$$

$$= \frac{(x+e^x)(1e^x + xe^x) - xe^x(1+e^x)}{(x+e^x)^2}$$

$$= \frac{\cancel{xe^x} + x^2e^x + e^{2x} + xe^{2x} - \cancel{xe^x} - xe^{2x}}{(x+e^x)^2}$$

$$= \boxed{\frac{x^2e^x + e^{2x} + xe^{2x} - xe^{2x}}{(x+e^x)^2}}$$

Example 9: Find an equation of the tangent line and normal line to the given curve $y = 2\sqrt{x}e^x + 1$ at the point $(0, 1)$.

$$y' = 2 \cdot \tfrac{1}{2} x^{-1/2} e^x + 2\sqrt{x}\, e^x$$

$$= e^x/\sqrt{x} + 2\sqrt{x}\, e^x$$

at $x=0$, $m = e^0/\sqrt{0} + 2\sqrt{0}\, e^0$ ← undefined. vertical tangent $\boxed{x=1}$

normal line has slope $\underline{\underline{0}}$, horizontal line $\boxed{y=0}$

Example 10: A manufacturer produces socks. The quantity q of these socks (measured in pairs of socks) that are sold are a function of the selling price p (in dollars), so we can write $q = f(p)$. Then the total revenue earned with a selling price p is $R(p) = pf(p)$.

(a) What does it mean to say $f(10) = 20,000$ and $f'(10) = 3,500$?

If the price is \$10, we sell 20,000 pairs of socks.

when the price is \$10, the number of socks sold is increasing at a rate of 3500 socks per \$1.

(b) Assuming the values in part (a), find $R'(10)$ and interpret your answer.

$$R'(p) = 1f(p) + pf'(p)$$

$$R'(10) = f(10) + 10 \cdot f'(10)$$

$$= 20,000 + 10(3500)$$

$$= 20,000 + 35,000$$

$$= \boxed{55,000}$$

when the price is \$10, revenue is increasing at a rate of \$55,000 per \$1 increase in price